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Answers and Explanations to BC Practice Exam 2

ANSWERS AND EXPLANATIONS TO SECTION 1

SECTION I, PART A

PROBLEM 1. What is the slope of the line tangent to the curve $x^2+2xy+3y^2=2$ when $y=1$?

We need to use implicit differentiation to find $\frac{dy}{dx}$.

$$2x+2\left(x\frac{dy}{dx}+y\right)+6y\frac{dy}{dx}=0$$

$$2x+2x\frac{dy}{dx}+2y+6y\frac{dy}{dx}=0$$

Now, if we wanted to solve for $\frac{dy}{dx}$ in terms of x and y, we would have to do some algebra to isolate $\frac{dy}{dx}$. But, because we are asked to solve for $\frac{dy}{dx}$ at a specific value of x, we don't need to simplify.

We need to find the x-coordinate that corresponds to the y-coordinate $y=1$. We plug $y=1$ into the equation and solve for x:

$$x^2+2x(1)+3(1)^2=2$$

$$x^2+2x+3=2$$

$$x^2+2x+1=0$$

$$(x+1)^2=0$$

$$x=-1$$

Finally, we plug $x=-1$ and $y=1$ into the derivative and we get:

$$2(-1)+2(-1)\frac{dy}{dx}+2(1)+6(1)\frac{dy}{dx}=0$$

$$-2-2\frac{dy}{dx}+2+6\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=0$$

The answer is (D).

PROBLEM 2. $\int_{-1}^{1} xe^{x^2}\,dx =$

We can use u-substitution to evaluate the integral.

Let $u = x^2$ and $du = 2xdx$. Then $\frac{1}{2}du = xdx$.

Now we substitute into the integral: $\frac{1}{2}\int e^u\,du$, leaving out the limits of integration for the moment.

Evaluate the integral to get: $\frac{1}{2}\int e^u\,du = \frac{1}{2}e^u$

Now we substitute back to get: $\frac{1}{2}e^{x^2}$

Finally, we evaluate at the limits of integration and we get: $\frac{1}{2}e^{x^2}\Big|_{-1}^{1} = \frac{1}{2}e - \frac{1}{2}e = 0$

The answer is (C).

PROBLEM 3. If, for $t > 0$, $x = t^2$ and $y = \cos\left(t^2\right)$, then $\frac{dy}{dx} =$

If we have a pair of parametric equations, $x(t)$ and $y(t)$, then $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

Here we get: $\frac{dy}{dt} = -2t\sin\left(t^2\right)$ and $\frac{dx}{dt} = 2t$

Then $\frac{dy}{dx} = \frac{-2t\sin\left(t^2\right)}{2t} = -\sin\left(t^2\right)$

The answer is (B).

Problem 4. The function $f(x)=4x^3-8x^2+1$ on the interval $[-1,1]$ has an absolute minimum at $x=$

If we want to find the minimum, we take the derivative and find where the derivative is zero.

$$f'(x)=12x^2-16x$$

Next, we set the derivative equal to zero and solve for x, in order to find the critical values.

$$12x^2-16x=0$$

$$4x(3x-4)=0$$

$$x=0 \text{ or } x=\frac{4}{3}.$$

Next, we can use the second derivative test to determine which critical value is a minimum and which is a maximum.

Remember the second derivative test: **If the sign of the second derivative at a critical value is positive, then the curve has a local minimum there. If the sign of the second derivative is negative, then the curve has a local maximum there.**

We take the second derivative: $f'(x)=24x-16$

This is negative at $x=0$ and positive at $x=\frac{4}{3}$. This means that the curve has a *relative* minimum at $x=\frac{4}{3}$, but, this value is outside of the interval $[-1,1]$. So, in order to find where it has an absolute *minimum*, we plug the endpoints of the interval into the original equation, and the smaller value will be the answer.

At $x=-1$, the value is $f(-1)=-11$. At $x=1$, the value is $f(1)=-3$.

The answer is (B).

Problem 5. $\int \frac{x\,dx}{x^2+5x+6}=$

Whenever we have an integrand that is a rational expression, we can often use the Method of Partial Fractions to rewrite the integral in a form where it's easy to evaluate.

First, separate the denominator into its two components and place the constants A and B in the numerators of the fractions and the components into the denominators. Set their sum equal to the original rational expression.

$$\frac{A}{x+3}+\frac{B}{x+2}=\frac{x}{(x+3)(x+2)}$$

Now we want to solve for the constants A and B. First, multiply through by $(x+3)(x+2)$ to clear the denominators.

$$(x+3)(x+2)\left[\frac{A}{x+3}+\frac{B}{x+2}\right]=x$$

$$A(x+2)+B(x+3)=x$$

Now distribute, then group, the terms on the left side.

$$Ax+2A+Bx+3B=x$$

$$Ax+Bx+2A+3B=x$$

$$(A+B)x+(2A+3B)=x$$

In order for this last equation to be true, we need $A + B = 1$ and $2A + 3B = 0$.

If we solve these simultaneous equations, we get: $A = 3$ and $B = -2$.

Now that we have done the partial fraction decomposition, we can rewrite the original integral as: $\int\left[\frac{3}{x+3}-\frac{2}{x+2}\right]dx$. This is now simple to evaluate.

$$\int\left[\frac{3}{x+3}-\frac{2}{x+2}\right]dx=3\int\frac{dx}{x+3}-2\int\frac{dx}{x+2}=3\ln|x+3|-2\ln|x+2|+C$$

Using the Rules of Logarithms, the answer can be rewritten as: $\ln\left|\frac{(x+3)^3}{(x+2)^2}\right|+C$

The answer is (A).

PROBLEM 6. $\frac{d}{dx}\left(x^2\sin^2 x\right)=$

Here we need to use the product rule, which is: If $f(x)=uv$, where u and v are both functions of x, then $f'(x)=u\frac{dv}{dx}+v\frac{du}{dx}$.

We get: $\frac{d}{dx}\left(x^2\sin^2 x\right)=2x\sin^2 x+x^2(2\sin x\cos x)$.

This can be simplified to: $2x\sin^2 x+x^2\sin 2x$

The answer is (E).

Problem 7. The line normal to the curve $y = \dfrac{x^2 - 1}{x^2 + 1}$ at $x = 2$ has slope

The normal line to a curve at a point is perpendicular to the tangent line at the same point. Thus, the slope of the normal line is the negative reciprocal of the slope of the tangent line. We find the slope of the tangent line by finding the derivative and evaluating it at the point.

We need to use the Quotient Rule, which is:

Given $y = \dfrac{g(x)}{h(x)}$ then $\dfrac{dy}{dx} = \dfrac{h(x)g'(x) - g(x)h'(x)}{[h(x)]^2}$

Here, we have: $\dfrac{dy}{dx} = \dfrac{(x^2+1)(2x) - (x^2-1)(2x)}{(x^2+1)^2}$

Next, Plug In $x = 2$ and solve: $\left.\dfrac{dy}{dx}\right|_2 = \dfrac{(4+1)(4) - (4-1)(4)}{(4+1)^2} = \dfrac{8}{25}$

Therefore, the slope of the normal line is $-\dfrac{25}{8}$.

The answer is (B).

Problem 8. If f and g are differentiable functions and $h(x) = f(x)e^{g(x)}$, then $h'(x) =$

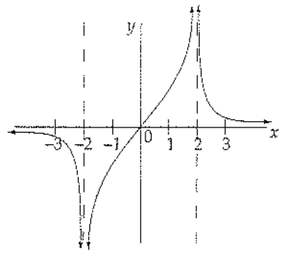
Here we need to use the product rule, which is: If $f(x) = uv$, where u and v are both functions of x, then $f'(x) = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$.

We get: $h'(x) = f'(x)e^{g(x)} + f(x)\left[e^{g(x)}g'(x)\right]$.

This can be simplified to: $h'(x) = e^{g(x)}\left[f'(x) + f(x)g'(x)\right]$

The answer is (C).

PROBLEM 9.



The graph of $y = f(x)$ is shown above. Which of the following could be the graph of $y = f'(x)$?

Here we want to examine the slopes of various pieces of the graph of $f(x)$. Notice that the graph starts with a slope of approximately zero and has a negative slope from $x = -\infty$ to $x = -2$, where the slope is $-\infty$. Thus we are looking for a graph of $f'(x)$ that is negative from $x = -\infty$ to $x = -2$ and undefined at $x = -2$. Next, notice that the graph of $f(x)$ has a positive slope from $x = -2$ to $x = 2$ and that the slope shrinks from ∞ to approximately one and then grows to ∞. Thus we are looking for a graph of $f'(x)$ that is positive from $x = -2$ to $x = 2$ and approximately equal to one at $x = 0$. Finally, notice that the graph of $f(x)$ has a negative slope from $x = 2$ to $x = \infty$, where the slope starts at $-\infty$ and grows to approximately zero. Thus we are looking for a graph of $f'(x)$ that is negative from $x = 2$ to $x = \infty$, where it is approximately zero. Graph (D) satisfies all of these requirements.

The answer is (D).

PROBLEM 10. $\int_e^{e^2} \left(\sqrt{x} + \frac{1}{\sqrt{x}} \right)^2 dx =$

First, expand the integrand: $\int_e^{e^2} \left(\sqrt{x} + \frac{1}{\sqrt{x}} \right)^2 dx = \int_e^{e^2} \left(x + 2 + \frac{1}{x} \right) dx$

Next, evaluate the integral:

$$\int_e^{e^2} \left(x + 2 + \frac{1}{x} \right) dx = \left(\frac{x^2}{2} + 2x + \ln|x| \right) \Bigg|_e^{e^2} = \left(\frac{e^4}{2} + 2e^2 + \ln e^2 \right) - \left(\frac{e^2}{2} + 2e + \ln e \right) = \frac{e^4}{2} + \frac{3e^2}{2} - 2e + 1$$

The answer is (D).

PROBLEM 11. $\int_4^{\infty} \frac{dx}{x^2+16}$

Whenever we have an integral of the form $\int \frac{dx}{a^2+x^2}$, where a is a constant, the integral is going to be an inverse tangent. So let's put the integrand into the desired form. Also, we're going to ignore the limits of integration until after we have done the antidifferentiation.

First divide the numerator and the denominator by 16: $\int \frac{dx}{x^2+16} = \frac{1}{16}\int \frac{dx}{\left(1+\frac{x^2}{16}\right)}$

Next, we do u-substitution. Let $u = \frac{x}{4}$ and $du = \frac{dx}{4}$ or $4du = dx$.

Substitute into the integrand: $\frac{1}{16}\int \frac{dx}{\left(1+\frac{x^2}{16}\right)} = \frac{1}{16}\int \frac{4du}{\left(1+u^2\right)} = \frac{1}{4}\int \frac{du}{\left(1+u^2\right)}$

Evaluate the integral: $\frac{1}{4}\int \frac{du}{\left(1+u^2\right)} = \frac{1}{4}\tan^{-1} u$

Substitute back to get: $\frac{1}{4}\tan^{-1}\left(\frac{x}{4}\right)$

Now we can evaluate the function at the limits of integration:

$$\left[\frac{1}{4}\tan^{-1}\left(\frac{x}{4}\right)\right]_4^{\infty} = \frac{1}{4}\left[\tan^{-1}\infty - \tan^{-1} 1\right] = \frac{1}{4}\left[\frac{\pi}{2} - \frac{\pi}{4}\right] = \frac{\pi}{16}$$

The answer is (A).

PROBLEM 12. What is the equation of the line tangent to the graph of $y = \sin^2 x$ at $x = \frac{\pi}{4}$?

If we want to find the equation of the tangent line, first we need to find the y-coordinate that corresponds to $x = \frac{\pi}{4}$. It is: $y = \sin^2\left(\frac{\pi}{4}\right) = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$

Next, we need to find the derivative of the curve at $x = \frac{\pi}{4}$, using the Chain Rule.

We get: $\frac{dy}{dx} = 2\sin x \cos x$. At $x = \frac{\pi}{4}$, $\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{4}} = 2\sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}\right) = 2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) = 1$

Now we have the slope of the tangent line and a point that it goes through. We can use the point-slope formula for the equation of a line, $(y - y_1) = m(x - x_1)$, and Plug In what we have just found. We get: $\left(y - \frac{1}{2}\right) = (1)\left(x - \frac{\pi}{4}\right)$

The answer is (B).

Problem 13. If $f(x) = \begin{cases} ax^2 + 3ax + 5\,;\ x \geq 2 \\ 4ax^3 - 6ax^2 + 9\,;\ x < 2 \end{cases}$, find the value of a that makes $f(x)$ continuous for all real values of x.

A polynomial is continuous everywhere on its domain, so we need to find a value of a such that $f(x)$ is continuous at $x = 2$. This means that $\lim_{x \to 2^-} f(x)$ must equal $\lim_{x \to 2^+} f(x)$. In other words, if we plug $x = 2$ into both pieces of this piecewise function, we need to get the same value:

$f(x) = \begin{cases} a(2)^2 + 3a(2) + 5\,;\ x \geq 2 \\ 4a(2)^3 - 6a(2)^2 + 9\,;\ x < 2 \end{cases}$, so we need $10a + 5 = 8a + 9$ and, therefore, $a = 2$

The answer is (D).

Problem 14. $\int x\sin(2x)dx =$

We can evaluate this integral using Integration By Parts. Here, we let $u = x$ and $dv = \sin(2x)\,dx$. Then $du = dx$ and $v = -\frac{1}{2}\cos(2x)$.

The rule for Integration By Parts says that $\int u\,dv = uv - \int v\,du$.

Substituting the terms we get: $\int x\sin(2x)dx = -\frac{x}{2}\cos(2x) + \frac{1}{2}\int \cos(2x)\,dx$.

Now, we integrate the second term, which gives us:

$$\int x\sin(2x)dx = -\frac{x}{2}\cos(2x) + \frac{1}{4}\sin(2x) + C\cdot$$

The answer is (C).

Problem 15. If $f(x) = \dfrac{x^2+5x-24}{x^2+10x+16}$, then $\lim\limits_{x\to-8} f(x)$ is

First, try plugging $x = -8$ into $f(x) = \dfrac{x^2+5x-24}{x^2+10x+16}$

We get: $f(x) = \dfrac{(-8)^2+5(-8)-24}{(-8)^2+10(-8)+16} = \dfrac{0}{0}$. This does NOT necessarily mean that the limit does not exist. When we get a limit of the form $\dfrac{0}{0}$, we first try to simplify the function by factoring and canceling like terms. Here we get:

$$f(x) = \frac{x^2+5x-24}{x^2+10x+16} = \frac{(x+8)(x-3)}{(x+8)(x+2)} = \frac{(x-3)}{(x+2)}$$

Now, if we Plug In $x = -8$, we get: $f(x) = \dfrac{(-8-3)}{(-8+2)} = \dfrac{-11}{-6} = \dfrac{11}{6}$

The answer is (D).

Problem 16. What is the approximation of the value of e^3 obtained by using the fourth-degree Taylor Polynomial about $x = 0$ for e^x?

The Taylor series for e^x about $x = 0$ is $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \ldots$

Here, we simply substitute 3 for x in the series and we get:

$$e^3 = 1 + 3 + \frac{3^2}{2!} + \frac{3^3}{3!} + \frac{3^4}{4!} + \ldots$$

The answer is (B).

PROBLEM 17. A rock is thrown straight upward with an initial velocity of $50\,m/s$ from a point $100\,m$ above the ground. If the acceleration of the rock at any time t is $a=-10\,m/s^2$, what is the maximum height of the rock (in meters)?

Because the derivative of velocity with respect to time is acceleration, we have:

$$v(t)=\int -10\,dt==-10t+C$$

Now we can Plug In the initial condition to solve for the constant:

$50=-10(0)+C$

$C=50$

Therefore, the velocity function is $v(t)=-10t+50$.

Note that the velocity is zero at $t=5$

Next, because the derivative of position with respect to time is velocity, we have:

$s(t)=\int(-10t+50)\,dt=-5t^2+50t+C$

Now we can Plug In the initial condition to solve for the constant:

$100=-5(0)^2+50(0)+C\ C=100$

Therefore, the position function is $s(t)=-5t^2+50t+100$.

The maximum height occurs when the velocity is zero, so we plug $t=5$ into the position function to get: $s(5)=-5(5)^2+50(5)+100=225$

The answer is (E).

PROBLEM 18. The sum of the infinite geometric series $2-\frac{2}{3}+\frac{2}{9}-\frac{2}{27}+\ldots$ is

This is the geometric series $\sum_{n=0}^{\infty}2\left(-\frac{1}{3}\right)^n$. The sum of an infinite series of the form

$\sum_{n=0}^{\infty}ar^n$ is $S=\frac{a}{1-r}$. Here, the sum is $S=\frac{2}{1-\left(-\frac{1}{3}\right)}=\frac{2}{\frac{4}{3}}=\frac{3}{2}$

The answer is (E).

PROBLEM 19. What are all values of x for which the series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n^2(5^n)}$ converges?

Use the Ratio Test to determine the interval of convergence.

We get: $$\lim_{n\to\infty} \frac{\dfrac{(x-3)^{n+1}}{(n+1)^2(5^{n+1})}}{\dfrac{(x-3)^n}{(n)^2(5^n)}} = \lim_{n\to\infty} \frac{(x-3)^{n+1}}{(n+1)^2(5^{n+1})} \frac{(n)^2(5^n)}{(x-3)^n} = \lim_{n\to\infty} \frac{x-3}{5} \frac{(n)^2}{(n+1)^2} = \frac{x-3}{5}$$

This converges if $\left|\frac{x-3}{5}\right| < 1$ or $-1 < \frac{x-3}{5} < 1$ and diverges if $\left|\frac{x-3}{5}\right| > 1$.

Thus the series converges when $-2 < x < 8$.

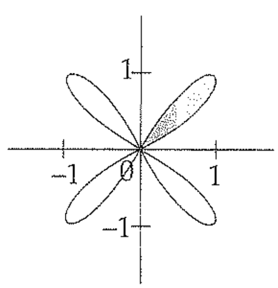
Now, we need to test whether the series converges at the endpoints of this interval.

When $x = 8$, we get the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which converges, and when $x = -2$, we get the alternating series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$, which also converges. Thus, the series converges when $-2 \le x \le 8$.

The answer is (A).

PROBLEM 20. Find the area inside one loop of the curve $r = \sin 2\theta$.

First, let's graph the curve:



Let's find the area of the loop in the first quadrant, which is the interval from $\theta = 0$ to $\theta = \frac{\pi}{2}$. We find the area of a polar graph by evaluating $A = \frac{1}{2}\int_0^{\frac{\pi}{2}} r^2 d\theta$.

Thus we need to evaluate $A=\frac{1}{2}\int_0^{\frac{\pi}{2}}\sin^2(2\theta)d\theta$.

Next, we need to do a trigonometric substitution to evaluate this integral. Recall that $\cos 2\theta=1-2\sin^2\theta$. We can rearrange this to obtain $\sin^2\theta=\frac{1-\cos 2\theta}{2}$, or in this case, $\sin^2 2\theta=\frac{1-\cos 4\theta}{2}$.

Thus, we can rewrite the integral: $A=\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{1-\cos(4\theta)}{2}d\theta$.

Now we can evaluate this integral:

$$\int\frac{1-\cos(4\theta)}{2}d\theta=\int\frac{d\theta}{2}-\int\frac{\cos(4\theta)}{2}d\theta=\frac{\theta}{2}-\frac{\sin(4\theta)}{8}$$

Now we evaluate the integral at the limits of integration:

$$\frac{1}{2}\left(\frac{\theta}{2}-\frac{\sin(4\theta)}{8}\right)\Bigg|_0^{\frac{\pi}{2}}=\frac{1}{2}\left(\frac{\pi}{4}-\frac{\sin(2\pi)}{8}\right)-\frac{1}{2}\left(0-\frac{\sin(0)}{8}\right)=\frac{\pi}{8}$$

The answer is (B).

Problem 21. The average value of $\sec^2 x$ on the interval $\left[\frac{\pi}{6},\frac{\pi}{4}\right]$ is

In order to find the average value, we use the Mean Value Theorem for Integrals, which says that the average value of $f(x)$ on the interval $[a,b]$ is $\frac{1}{b-a}\int_a^b f(x)\,dx$.

Here, we have $\frac{1}{\frac{\pi}{4}-\frac{\pi}{6}}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\sec^2 x\,dx$.

Next, recall that $\frac{d}{dx}\tan x=\sec^2 x$.

We evaluate the integral: $\frac{1}{\frac{\pi}{4}-\frac{\pi}{6}}(\tan x)\Big|_{\frac{\pi}{6}}^{\frac{\pi}{4}}=\frac{1}{\frac{\pi}{4}-\frac{\pi}{6}}\left[\tan\frac{\pi}{4}-\tan\frac{\pi}{6}\right]=\frac{1}{\frac{\pi}{4}-\frac{\pi}{6}}\left(1-\frac{\sqrt{3}}{3}\right)$

Next, we need to do a little algebra. Get a common denominator for each of the two expressions: $\dfrac{1}{\frac{\pi}{4}-\frac{\pi}{6}}\left(1-\dfrac{\sqrt{3}}{3}\right)=\dfrac{1}{\frac{6\pi}{24}-\frac{4\pi}{24}}\left(\dfrac{3}{3}-\dfrac{\sqrt{3}}{3}\right)$

We can simplify this to: $\dfrac{1}{\frac{2\pi}{24}}\left(\dfrac{3-\sqrt{3}}{3}\right)=\dfrac{12}{\pi}\left(\dfrac{3-\sqrt{3}}{3}\right)=\dfrac{12-4\sqrt{3}}{\pi}$

The answer is (C).

Problem 22. Find the length of the arc of the curve defined by $x=\frac{1}{2}t^2$ and $y=\frac{1}{9}(6t+9)^{\frac{3}{2}}$, from $t = 0$ to $t = 2$.

We can find the length of a parametric curve on the interval $[a,b]$, by evaluating the integral $\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$

First, we take the derivatives $\frac{dx}{dt}=t$ and $\frac{dy}{dt}=\sqrt{6t+9}$

Now, square the derivatives $\left(\frac{dx}{dt}\right)^2=t^2$ and $\left(\frac{dy}{dt}\right)^2=6t+9$

Now, we plug this into the formula and we get:

$$\int_0^2 \sqrt{t^2+6t+9}\,dt=\int_0^2 \sqrt{(t+3)^2}\,dt=\int_0^2 (t+3)\,dt=\left(\frac{t^2}{2}+3t\right)\Bigg|_0^2=8$$

The answer is (A).

Problem 23. The function f is given by $f(x)=x^4+4x^3$. On which of the following intervals is f decreasing?

A function is decreasing on an interval where the derivative is negative.

The derivative is $f'(x)=4x^3+12x^2$

Next, we want to determine on which intervals the derivative of the function is positive and on which it is negative. We do this by finding where the derivative is zero:

$4x^3 + 12x^2 = 0$

$4x^2(x+3) = 0$

$x = -3$ or $x = 0$

We can test where the derivative is positive and negative by picking a point in each of the three regions $-\infty < x < -3$, $-3 < x < 0$, and $0 < x < \infty$, plugging the point into the derivative, and seeing what the sign of the answer is. You should find that the derivative is negative on the interval $-\infty < x < -3$.

The answer is (D).

Problem 24. Which of the following series converge(s)?

I. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ II. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3}}$ III. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2}}$

Series I: You might recognize this as the alternating harmonic series, which converges. If you don't, use the alternating series test, which says that, in order for an alternating series $\sum_{n=1}^{\infty} (-1)^n a_n$ to converge: (1) $a_{n+1} < a_n$ and (2) $\lim_{n \to \infty} a_n = 0$. This series satisfies both conditions, so it converges.

Series II: We can rewrite this series as $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$, which is a p-series. A p-series is a series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$, which converges if $p > 1$ and diverges if $p < 1$. Thus, this series converges.

Series III: We can rewrite this series as $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{2}{3}}}$, which is also a p-series. Because $p < 1$, this series diverges.

The answer is (C).

Problem 25. Given the differential equation $\frac{dz}{dt} = z\left(4 - \frac{z}{100}\right)$, where $z(0) = 50$, what is $\lim_{t \to \infty} z(t)$?

We can solve this differential equation by separation of variables.

$$\int \frac{dz}{z\left(4 - \frac{z}{100}\right)} = \int dt$$

The integral on the right is trivial. We get $t + C$.

The one on the left will require the Method of Partial Fractions. First, separate the denominator into its two components and place the constants A and B in the numerators of the fractions and the components into the denominators. Set their sum equal to the original rational expression $\frac{A}{z} + \frac{B}{4 - \frac{z}{100}} = \frac{1}{z\left(4 - \frac{z}{100}\right)}$

Now we want to solve for the constants A and B. First, multiply through by $z\left(4 - \frac{z}{100}\right)$ to clear the denominators.

$$z\left(4 - \frac{z}{100}\right)\left[\frac{A}{z} + \frac{B}{\left(4 - \frac{z}{100}\right)}\right] = 1$$

$$A\left(4 - \frac{z}{100}\right) + Bz = 1$$

Now distribute, then group, the terms on the left side.

$$4A - A\frac{z}{100} + Bz = 1$$

$$z\left(B - \frac{A}{100}\right) + 4A = 1$$

In order for this last equation to be true, we need $4A = 1$ and $B - \frac{A}{100} = 0$.

If we solve these simultaneous equations, we get: $A = \frac{1}{4}$ and $B = \frac{1}{400}$.

Now that we have done the partial fraction decomposition, we can rewrite the original integral as: $\int \frac{\frac{1}{4}}{z} + \frac{\frac{1}{400}}{\left(4 - \frac{z}{100}\right)} dz$. This is now simple to evaluate.

$$\int \frac{\frac{1}{4}}{z} + \frac{\frac{1}{400}}{\left(4 - \frac{z}{100}\right)} dz = \frac{1}{4}\int \frac{dz}{z} + \frac{1}{400}\int \frac{dz}{\left(4 - \frac{z}{100}\right)} = \frac{1}{4}\ln|z| - \frac{1}{4}\ln\left|4 - \frac{z}{100}\right| + C$$

We can rewrite this with the laws of logarithms to get:

$$\frac{1}{4}\ln|z| - \frac{1}{4}\ln\left|4 - \frac{z}{100}\right| = \ln\left(\frac{100z}{400 - z}\right)^{\frac{1}{4}}$$

Thus, the solution to the differential equation is $\ln\left(\frac{100z}{400 - z}\right)^{\frac{1}{4}} = t + C$. We will need to do some algebra to rearrange the equation. First, exponentiate both sides to base e:

$$\left(\frac{100z}{400 - z}\right)^{\frac{1}{4}} = e^{t+C}$$

Then, because $e^{t+C} = e^t e^C$, and because e^c is a constant we get: $\left(\frac{100z}{400 - z}\right)^{\frac{1}{4}} = Ce^t$

Next, raise both sides to the fourth power: $\left(\frac{100z}{400 - z}\right) = Ce^{4t}$

Invert both sides: $\left(\frac{400 - z}{100z}\right) = Ce^{-4t}$

Break the left side into two fractions: $\frac{4}{z} - \frac{1}{100} = Ce^{-4t}$

Add $\frac{1}{100}$ to both sides: $\frac{4}{z} = \frac{1}{100} + Ce^{-4t} = \frac{Ce^{-4t} + 1}{100}$

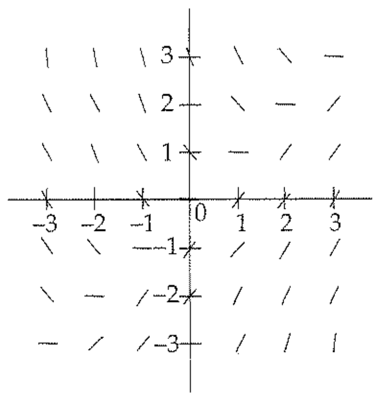
Invert both sides (again): $\frac{z}{4} = \frac{100}{Ce^{-4t} + 1}$

Finally: $z(t)=\dfrac{400}{Ce^{-4t}+1}$ (Whew!)

Now, we can take the limit: $\lim\limits_{t\to\infty}\dfrac{400}{Ce^{-4t}+1}=400$

The answer is (A).

PROBLEM 26.



The slope field shown above corresponds to which of the following differential equations?

Notice that the slope of the differential equation is zero (horizontal tangent) at the origin. This eliminates answer choices (A) and (B) because they are undefined at the origin (so they would show a vertical tangent there). Next, notice that the slope is positive at $(1,0)$. This eliminates answer choice (C), which is zero on both axes. Finally, notice that the slope is negative at $(0,1)$, which eliminates answer choice (E), which is positive there.

The answer is (D).

PROBLEM 27. The value of c that satisfies the Mean Value Theorem for Derivatives on the interval $[0,5]$ for the function $f(x)=x^3-6x$ is

The Mean Value Theorem for Derivatives says that, given a function $f(x)$ which is continuous and differentiable on $[a,b]$, then there exists some value c on (a,b) where

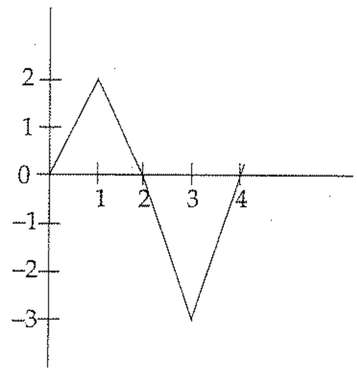
$$\frac{f(b)-f(a)}{b-a}=f'(c)$$

Here, we have $\dfrac{f(b)-f(a)}{b-a}=\dfrac{f(5)-f(0)}{5-0}=\dfrac{95-0}{5}=19$

and $f'(c)=3c^2-6$, so we simply set $3c^2-6=19$. If we solve for c, we get: $c=\pm\frac{5}{\sqrt{3}}$. Both of these values satisfy the Mean Value Theorem for Derivatives, but only the positive value, $c=\frac{5}{\sqrt{3}}$, is in the interval.

The answer is (E).

PROBLEM 28.



The graph of f is shown in the figure above. If $g(x)=\int_0^x f(t)dt$, for what positive value of x does $g(x)$ have a minimum?

If we want to find where $g(x)$ is a minimum, we can look at $g'(x)$. The Second Fundamental Theorem of Calculus tells us how to find the derivative of an integral: $\frac{d}{dx}\int_c^x f(t)\,dt=f(x)$, where c is a constant. Thus, $g'(x)=f(x)$. The graph of f is zero at $x=0$, $x=2$, and $x=4$. We can eliminate $x=0$ because we are looking for a *positive* value of x. Next, notice that f is negative to the left of $x=4$ and positive to the right of $x=4$. Thus, $g(x)$ has a minimum at $x=4$.

We also could have found the answer geometrically. The function $g(x)=\int_0^x f(t)\,dt$ is called an *accumulation function* and stands for the area between the curve and the x-axis to the point x. Thus, the value of g grows from $x=0$ to $x=2$. Then, because we subtract the area under the x-axis from the area above it, the value of g shrinks from $x=2$ to $x=4$. The value begins to grow again after $x=4$.

The answer is (E).

Problem 29. If $f(x)$ is the function given by $f(x)=e^{3x}+1$, at what value of x is the slope of the tangent line to $f(x)$ equal to 2?

The slope of the tangent line is the derivative of the function. We get: $f'(x)=3e^{3x}$ Now we set the derivative equal to 2 and solve for x.

$$3e^{3x}=2$$

$$e^{3x}=\frac{2}{3}$$

$$3x=\ln\frac{2}{3}$$

$$x=\frac{1}{3}\ln\frac{2}{3}\approx -.135$$

(Remember to round all answers to three decimal places on the AP exam).

The answer is (A).

Problem 30. If $y=(\sin x)^{e^x}$, then $y'=$

We need to use logarithmic differentiation to find the derivative. First, take the log of both sides: $\ln y=\ln\left[(\sin x)^{e^x}\right]$

Next, on the right side, put the power in front of the log: $\ln y=e^x\ln(\sin x)$

Next, take the derivative of both sides: $\frac{1}{y}\frac{dy}{dx}=e^x\ln(\sin x)+e^x\frac{\cos x}{\sin x}$, which can be simplified to: $\frac{1}{y}\frac{dy}{dx}=e^x\ln(\sin x)+e^x\cot x$.

Multiply both sides by y: $\frac{dy}{dx}=y\left[e^x\ln(\sin x)+e^x\cot x\right]$.

Substitute $y=(\sin x)^{e^x}$ for y on the right side: $\frac{dy}{dx}=(\sin x)^{e^x}\left[e^x\ln(\sin x)+e^x\cot x\right]$.

Finally, factor out e^x to obtain: $\frac{dy}{dx}=e^x(\sin x)^{e^x}\left[\ln(\sin x)+\cot x\right]$.

The answer is (D).

Problem 31. The side of a square is increasing at a constant rate of 0.4 $cm\ /\ sec$. In terms of the perimeter, P, what is the rate of change of the area of the square, in $cm^2\ /\ sec$?

The formula for the perimeter of a square is $P = 4s$, where s is the length of a side of the square.

If we differentiate this with respect to t, we get $\frac{dP}{dt} = 4\frac{ds}{dt}$. We Plug In $\frac{ds}{dt} = 0.4$ and we get $\frac{dP}{dt} = 4(0.4) = 1.6$

The formula for the area of a square is $A = s^2$. If we solve the perimeter equation for s in terms of P and substitute it into the area equation we get:

$$s = \frac{P}{4}, \text{ so } A = \left(\frac{P}{4}\right)^2 = \frac{P^2}{16}$$

If we differentiate this with respect to t, we get $\frac{dA}{dt} = \frac{P}{8}\frac{dP}{dt}$

Now we Plug In $\frac{dP}{dt} = 1.6$ and we get: $\frac{dA}{dt} = \frac{P}{8}(1.6) = 0.2P$.

The answer is (B).

Problem 32. If f is a vector-valued function defined by $f(t) = (\sin 2t, \sin^2 t)$, then $f''(t) =$

The acceleration vector is the second derivative of the position vector (The velocity vector is the first derivative.).

The velocity vector of this particle is: $(2\cos(2t), 2\sin(t)\cos(t))$, which can be simplified to: $(2\cos(2t), \sin(2t))$.

The acceleration vector is: $(-4\sin(2t), 2\cos(2t))$.

The answer is (A).

Problem 33. The height of a mass hanging from a spring at time t seconds, where $t > 0$, is given by $h(t) = 12 - 4\cos(2t)$. In the first two seconds, how many times is the velocity of the mass equal to 0?

The velocity of the mass is the first derivative of the height: $v(t) = 8\sin(2t)$.

Now, graph the equation to find how many times the graph of $v(t)$ crosses the t-axis between $t = 0$ and $t = 2$. Or you should know that this is a sine graph with an amplitude of 8 and a period of π, which will cross the t-axis once on the interval at

$t = \frac{\pi}{2}$.

The answer is (B).

PROBLEM 34. $\lim\limits_{h \to 0} \frac{\tan^{-1}(1+h) - \frac{\pi}{4}}{h} =$

Notice how this limit takes the form of the definition of the Derivative, which is:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Here, if we think of $f(x)$ as $\tan^{-1} x$, then this expression gives the derivative of $\tan^{-1} x$ at the point $x = \frac{\pi}{4}$.

The derivative of $\tan^{-1} x$ is $f'(x) = \frac{1}{1+x^2}$. At $x = \frac{\pi}{4}$, we get

$$f'\left(\frac{\pi}{4}\right) = \frac{1}{1+\left(\frac{\pi}{4}\right)^2} = \frac{1}{1+\frac{\pi^2}{16}} = \frac{1}{\frac{16+\pi^2}{16}} = \frac{16}{16+\pi^2}$$

The answer is (C).

PROBLEM 35. What is the trapezoidal approximation of $\int_0^3 e^x dx$ using $n = 4$ subintervals?

The Trapezoid Rule enables us to approximate the area under a curve with a fair degree of accuracy. The rule says that the area between the x-axis and the curve $y = f(x)$, on the interval $[a, b]$, with n trapezoids, is:

$$\frac{1}{2}\frac{b-a}{n}\left[y_0 + 2y_1 + 2y_2 + 2y_3 + \ldots + 2y_{n-1} + y_n\right]$$

Using the rule here, with $n = 4$, $a = 0$, and $b = 3$, we get:

$$\frac{1}{2}\cdot\frac{3}{4}\left[e^0 + 2e^{\frac{3}{4}} + 2e^{\frac{6}{4}} + 2e^{\frac{9}{4}} + e^3\right] \approx 19.972$$

The answer is (C).

Problem 36. Given $x^2y + x^2 = y^2 + 1$, find $\frac{d^2y}{dx^2}$ at $(1,1)$.

Use implicit differentiation to find $\frac{dy}{dx}$: $x^2\frac{dy}{dx} + 2xy + 2x = 2y\frac{dy}{dx}$

Now we want to isolate $\frac{dy}{dx}$, which will take some algebra.

First, put all of the terms containing $\frac{dy}{dx}$ on one side of the equals sign and all of the other terms on the other side: $2xy + 2x = 2y\frac{dy}{dx} - x^2\frac{dy}{dx}$

Next, factor $\frac{dy}{dx}$ out of the right hand side: $2xy + 2x = \frac{dy}{dx}\left(2y - x^2\right)$

Finally, divide both sides by $\left(2y - x^2\right)$ to isolate $\frac{dy}{dx}$: $\frac{dy}{dx} = \frac{2xy + 2x}{2y - x^2}$

At $(1,1)$, we get: $\frac{dy}{dx} = \frac{2+2}{2-1} = 4$

Now, we can find the second derivative by again performing implicit differentiation:

$$\frac{d^2y}{dx^2} = \frac{\left(2y - x^2\right)\left(2x\frac{dy}{dx} + 2y + 2\right) - \left(2xy + 2x\right)\left(2\frac{dy}{dx} - 2x\right)}{\left(2y - x^2\right)^2}$$

At $(1,1)$, we get: $\frac{d^2y}{dx^2} = \frac{(2-1)(8+2+2)-(2+2)(8-2)}{(2-1)^2} = \frac{12-24}{1} = -12$

The answer is (D).

Problem 37. If $\int_{-2}^{4} f(x)dx = a$ and $\int_{3}^{4} f(x)dx = b$ then $\int_{3}^{-2} f(x)\,dx =$

Because the integral of a function can be interpreted as the area between the function and the curve, we can say that $\int_{a}^{b} f(x)\,dx = \int_{a}^{c} f(x)\,dx + \int_{c}^{b} f(x)\,dx$, where c is a point in the interval (a,b). Here we have: $\int_{-2}^{4} f(x)dx = \int_{-2}^{3} f(x)dx + \int_{3}^{4} f(x)dx$. We can rearrange this to: $\int_{-2}^{4} f(x)dx - \int_{3}^{4} f(x)dx = \int_{-2}^{3} f(x)dx$. This means that $a - b = \int_{-2}^{3} f(x)dx$. Finally, we know that $\int_{a}^{b} f(x)dx = -\int_{b}^{a} f(x)dx$, so $b - a = \int_{3}^{-2} f(x)dx$.

The answer is (D).

Problem 38. $\frac{d}{dx}\int_{2x}^{5x} \cos t\,dt =$

The Second Fundamental Theorem of Calculus tells us how to find the derivative of an integral: $\frac{d}{dx}\int_{v}^{u} f(t)\,dt = f(u)\frac{du}{dx} - f(v)\frac{dv}{dx}$, where u and v are functions of x.

Here we can use the theorem to get: $\frac{d}{dx}\int_{2x}^{5x} \cos t\,dt = 5\cos 5x - 2\cos 2x$.

The answer is (A).

Problem 39. Using the Taylor Series about $x = 0$ for sin x, approximate $\sin(0.4)$ to four decimal places.

We are given that $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$

Here, we simply substitute 0.4 for x in the series and we get:

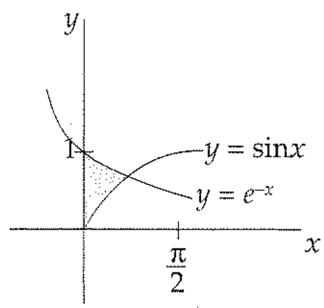
$$\sin 0.4 = 0.4 - \frac{(0.4)^3}{3!} + \frac{(0.4)^5}{5!} + \ldots$$

Find the value on your calculator. Keep adding terms until you get four decimal places of accuracy. You should get: $\sin(0.4) = 0.3894$ If you only got 0.3893, you didn't use enough terms (you need to use the fifth power term).

The answer is (B).

Problem 40. Let R be the region in the first quadrant between the graphs of $y = e^{-x}$, $y = \sin x$, and the y-axis. The volume of the solid that results when R is revolved about the x-axis is

First, we graph the curves.



We can find the volume by taking a vertical slice of the region . The formula for the volume of a solid of revolution around the y-axis, using a vertical slice bounded from above by the curve $f(x)$ and from below by $g(x)$, on the interval $[a,b]$, is:

$$\pi\int_a^b \left[f(x)^2 - g(x)^2\right]dx$$

The upper curve is $y = e^{-x}$ and the lower curve is $y = \sin x$.

Next, we need to find the point(s) of intersection of the two curves with a calculator (Good luck doing it by hand!), which we do by setting them equal to each other and solving for x. You should get approximately: $x = 0.589$ (remember to round to three decimal places on the AP.

Thus, the limits of integration are $x = 0$ and $x = 0.589$.

Now, we evaluate the integral:

$$\pi\int_0^{0.589}\left[\left(e^{-x}\right)^2 - \left(\sin^2 x\right)\right]dx = \pi\int_0^{0.589}\left(e^{-2x} - \sin^2 x\right)dx$$

We can evaluate this integral by hand but, because we will need a calculator to find the answer, we might as well use it to evaluate the integral.

We get: $\pi\int_0^{0.589}\left(e^{-2x}-\sin^2 x\right)dx = 0.888$ (rounded to three decimal places).

The answer is (E).

Problem 41. Use Euler's method, with $h = 0.2$ to estimate $y(3)$, if $\frac{dy}{dx} = 2y - 4x$ and $y(2) = 6$.

We can use Euler's Method to find an approximate answer to the differential equation. The method is quite simple. First, we need a starting point, (x_0, y_0) and an initial slope, y_0'. Next, we use increments of h to come up with approximations. Each new approximation will use the following rules:

$$x_n = x_{n-1} + h$$

$$y_n = y_{n-1} + h \cdot y'_{n-1}$$

Repeat for $n = 1, 2, 3, \ldots$

We are given that the curve goes through the point $(2,6)$. We will call the coordinates of this point $x_0 = 2$ and $y_0 = 6$. The slope is found by plugging these coordinates into $y' = 2y - 4x$, so we have an initial slope of $y_0' = 4$.

Now we need to find the next set of points.

Step 1: Increase x_0 by h to get x_1. $x_1 = 2.2$

Step 2: Multiply h by y_0' and add to y_0 to get y_1. $y_1 = 6 + 0.2(4) = 6.8$

Step 3: Find y_1' by plugging x_1 and y_1 into the equation for y'

$$y_1' = 2(6.8) - 4(2.2) = 4.8$$

Repeat until we get to the desired point (in this case $x = 3$).

Step 1: Increase x_1 by h to get x_2. $x_2 = 2.4$

Step 2: Multiply h by y_1' and add to y_1 to get y_2. $y_2 = 6.8 + 0.2(4.8) = 7.76$

Step 3: Find y_2' by plugging x_2 and y_2 into the equation for y'

$$y_2' = 2(7.76) - 4(2.4) = 5.92$$

Step 1: $x_3 = x_2 + h$. $\quad x_3 = 2.6$

Step 2: $y_3 = y_2 + h(y_2')$ $\quad y_3 = 7.76 + 0.2(5.92) = 8.944$

Step 3: $y_3' = 2(y_3) - 4(x_3)$ $\quad y_3' = 2(8.944) - 4(2.6) = 7.488$

Step 1: $x_4 = x_3 + h$. $\quad x_4 = 2.8$

Step 2: $y_4 = y_3 + h(y_3')$ $\quad y_4 = 8.944 + 0.2(7.488) = 10.4416$

Step 3: $y_4' = 2(y_4) - 4(x_4)$ $\quad y_4' = 2(10.4416) - 4(2.8) = 9.6832$

Step 1: $x_5 = x_4 + h$. $\quad x_5 = 3$

Step 2: $y_5 = y_4 + h(y_4')$ $\quad y_5 = 10.4416 + 0.2(9.6832) = 12.378$

The answer is (C).

PROBLEM 42. $\int \sec^4 x\, dx =$

First, break up the integrand: $\int \sec^4 x\, dx = \int (\sec^2 x)(\sec^2 x)dx$

Next, use the trig identity $1 + \tan^2 x = \sec^2 x$ to rewrite the integral:

$$\int (\sec^2 x)(\sec^2 x)dx = \int (1 + \tan^2 x)(\sec^2 x)dx = \int (\sec^2 x + \sec^2 x \tan^2 x)dx$$

We can evaluate these integrals separately.

The left one is easy: $\int \sec^2 x\, dx = \tan x + C$

We will use u-substitution for the right one. Let $u = \tan x$ and $du = \sec^2 x\, dx$. Then substitute into the integral and integrate: $\int \sec^2 x \tan^2 x\, dx = \int u^2 du = \frac{1}{3}u^3 + C$

Now substitute back: $\frac{1}{3}\tan^3 x + C$

Combine the two integrals to get: $\tan x + \frac{1}{3}\tan^3 x + C$

The answer is (B).

Problem 43. Let $f(x)=\int \cot x\,dx$; $0<x<\pi$. If $f\left(\frac{\pi}{6}\right)=1$, then $f(1)=$

We find $\int \cot x\,dx$ by rewriting the integral as $\int \frac{\cos x}{\sin x}dx$. Then we use u-substitution. Let u = sin$x$ and du = cos$x$. Substituting, we can get:

$$\int \frac{\cos x}{\sin x}dx=\int \frac{du}{u}=\ln|u|+C.$$

Then substituting back, we get: $\ln(\sin x)+C$ (We can get rid of the absolute value bars because sine is always positive on the interval.). Next, we use $f\left(\frac{\pi}{6}\right)=1$ to solve for C.

We get: $1=\ln\left(\sin\frac{\pi}{6}\right)+C$

$1=\ln\left(\frac{1}{2}\right)+C$

$1-\ln\left(\frac{1}{2}\right)=C=1.693147$

Thus, $f(x)=\ln(\sin x)+1.693147$

At x = 1, we get $f(1)=\ln(\sin 1)+1.693147=1.521$ (rounded to three decimal places).

The answer is (E).

Problem 44. $\int \sqrt{4-x^2}\,dx=$

We solve an integral of the form $\sqrt{a^2-x^2}$ by performing the trig substitution $x=a\sin\theta$. Here we will use $x=2\sin\theta$, which means that $dx=2\cos\theta\,d\theta$. We get: $\int\left[\sqrt{4-4\sin^2\theta}(2\cos\theta)\right]d\theta$. Next, factor out the 4: $\int 4\sqrt{1-\sin^2\theta}\cos\theta\,d\theta$.

Next, simplify the radicand: $\int 4\sqrt{\cos^2\theta}\cos\theta\,d\theta$, which gives us: $4\int \cos^2\theta\,d\theta$. Now we need to use another trig identity. Recall that $\cos 2\theta=2\cos^2\theta-1$, which can be rewritten as $\cos^2\theta=\frac{1+\cos 2\theta}{2}$.

Now we can rewrite the integral: $4\int \cos^2\theta \, d\theta = 4\int \frac{1+\cos 2\theta}{2} d\theta = 2\int (1+\cos 2\theta)\, d\theta$

Integrate: $2\int (1+\cos 2\theta)\, d\theta = 2\theta + \sin 2\theta + C$. Now we have to substitute back.

Because $x = 2\sin\theta$, we know that $\theta = \sin^{-1}\left(\frac{x}{2}\right)$ and that $\cos\theta = \frac{\sqrt{4-x^2}}{2}$.

This gives us: $2\theta + \sin 2\theta + C = 2\sin^{-1}\left(\frac{x}{2}\right) + \frac{x}{2}\sqrt{4-x^2} + C$.

The answer is (C).

Problem 45: A force of 250 N is required to stretch a spring 5 m from rest. Using Hooke's law, $F = kx$, how much work, in *Joules*, is required to stretch the spring 7 m from rest?

Hooke's law says that the force needed to compress or stretch a spring from its natural state is $F = kx$, where k is the spring constant. We can find the value of k from the initial information, namely $F = 250\ N$ for a stretch of 5 m. Thus, we can solve to find the value of k : $k = 250/5 = 50\ N/m$.

We find the work done by a variable force along the x-axis from $x = a$ to $x = b$ by evaluating the integral for the Work, $W = \int_a^b F(x)dx$.

Using the information we have $W = \int_a^b F(x)dx = \frac{1}{2}kx^2 |_0^7 = \frac{1}{2}50(7)^2 = 1225N$.

The correct answer is (E).

ANSWERS AND EXPLANATIONS TO SECTION II

Problem 1: An object moving along a curve in the xy-plane has its position given by $(x(t), y(t))$ at time t seconds, $0 \le t \le 1$, with $\frac{dx}{dt} = 8t\cos t$ *units / sec* and $\frac{dy}{dt} = 8t\sin t$ *units / sec*.

At time $t = 0$, the object is located at $(5,11)$

(a) Find the speed of the object at time $t = 1$.

(b) Find the length of the arc described by the curve's position from time $t = 0$ to time $t = 1$.

(c) Find the location of the object at time $t = \frac{\pi}{2}$.

(a) The equation for the speed of an object is $speed = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

Here, we get: $speed = \sqrt{(8t\cos t)^2 + (8t\sin t)^2}$

This can be simplified to: $speed = \sqrt{64t^2\cos^2 t + 64t^2\sin^2 t} = 8t\sqrt{\cos^2 t + \sin^2 t} = 8t$.

Thus, the speed of the object at time $t = 1$ is 8.

(b) We can find the length of a parametric curve $(x(t), y(t))$, on the interval $[a, b]$, by evaluating the integral $\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

We found the integrand in part (a): $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = 8t$. Thus, all we have to do is integrate this with respect to t from $t = 0$ to $t = 1$.

We get: $\int_0^1 8t\, dt = \left(4t^2\right)\Big|_0^1 = 4$

(c) First, we will need to find the equations for the coordinates, which we can do by solving the differential equations. First, let's find x: $\frac{dx}{dt} = 8t\cos t$.

We can solve this by separation of variables: First, we move dt to the right side of the equals sign: $dx = 8t\cos t\, dt$

Integrate both sides (pulling the constant out of the integrand): $\int dx = 8\int t\cos t\, dt$

The integral on the left is trivial: $\int dx = x$.

We will need to use integration by parts to solve the integral on the right.

The rule for Integration By Parts says that: $\int u\, dv = uv - \int v\, du$

Here, we let $u = t$ and $dv = \cos t\, dt$. Then $du = dt$ and $v = \sin t$

Substituting the terms we get: $8\int t\cos t\, dt = 8t\sin t - 8\int \sin t dt$

Now, we integrate the second term, which gives us: $8\int t\cos t\, dt = 8t\sin t + 8\cos t + C$

Thus, the x-coordinate is: $x(t) = 8t\sin t + 8\cos t + C$

Now, we Plug In the initial condition: $5 = (8)(0)\sin(0) + (8)\cos(0) + C$

This means that $C = -3$, so the equation for the x-coordinate is:

$$x(t) = 8t\sin t + 8\cos t - 3$$

Now let's find y: $\frac{dy}{dt} = 8t\sin t$

First, move dt to the right side of the equals sign: $dy = 8t\sin t\, dt$

Integrate both sides (pulling the constant out of the integrand): $\int dy = 8\int t\sin t\, dt$

The integral on the left is trivial: $\int dy = y$.

Again, we will need to use Integration By Parts to solve the integral on the right.

Here, we let $u = t$ and $dv = \sin t\, dt$. Then $du = dt$ and $v = -\cos t$.

Substituting the terms we get: $8\int t\sin t\, dt = -8t\cos t + 8\int \cos t dt$

Now, we integrate the second term, which gives us: $8\int t\sin t\, dt = -8t\cos t + 8\sin t + C$

Thus, the y-coordinate is: $y(t) = -8t\cos t + 8\sin t + C$

Now, we Plug In the initial condition: $11 = -(8)(0)\cos(0) + (8)\sin(0) + C$

This means that $C = 11$, so the equation for the y-coordinate is:

$$y(t) = -8t\cos t + 8\sin t + 11$$

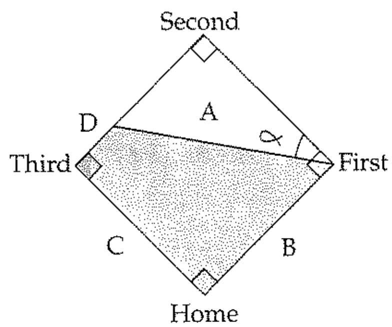
Finally, we plug $t = \frac{\pi}{2}$ into the equations for the coordinates:

$$x\left(\frac{\pi}{2}\right) = (8)\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right) + (8)\cos\left(\frac{\pi}{2}\right) - 3 = 4\pi - 3$$

$$y\left(\frac{\pi}{2}\right) = -(8)\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right) + (8)\sin\left(\frac{\pi}{2}\right) + 11 = 19$$

Therefore, the object's location at time $t = \frac{\pi}{2}$ is $(4\pi - 3, 19)$.

Problem 2.



A baseball diamond is a square with each side 90 feet in length. A player runs from second base to third base at a rate of 18 ft/sec.

(a) At what rate is the player's distance from first base, A, changing when his distance from third base, D, is 22.5 feet?

(b) At what rate is angle α increasing when D is 22.5 feet?

(c) At what rate is the area of the trapezoidal region, formed by line segments A, B, C, and D, changing when D is 22.5 feet?

(a) A is related to D by the Pythagorean Theorem: $90^2 + (90 - D)^2 = A^2$. This can be simplified to: $16200 - 180D + D^2 = A^2$

Take the derivative of both sides with respect to t: $-180\frac{dD}{dt} + 2D\frac{dD}{dt} = 2A\frac{dA}{dt}$

Now, we are given that $\frac{dD}{dt} = -18$ (It's negative because D is shrinking.) and $D = 22.5$.

Next, we need to solve for A: $90^2 + (90 - 22.5)^2 = A^2$. You should get $A = 112.5$

Now we can Plug In and solve for $\frac{dA}{dt}$:

$$-180(-18) + 2(22.5)(-18) = 2(112.5)\frac{dA}{dt}$$

$$3240 - 810 = 225\frac{dA}{dt}$$

$$\frac{dA}{dt} = 10.8 \; ft/sec$$

(b) Notice that $\tan\alpha = \frac{90-D}{90} = 1 - \frac{D}{90}$. We differentiate both sides with respect to t:

$$\sec^2\alpha \frac{d\alpha}{dt} = -\frac{1}{90}\frac{dD}{dt}$$

Next, we need to solve for $\sec^2\alpha$ when $D = 22.5$. From part (a), we know that A = 112.5, so $\sec\alpha = \frac{112.5}{90}$, so $\sec^2\alpha = \frac{25}{16}$.

Now we Plug In to solve for $\frac{d\alpha}{dt}$: $\left(\frac{25}{16}\right)\frac{d\alpha}{dt} = -\frac{1}{90}(-18)$

$\frac{d\alpha}{dt} = \frac{16}{125} = 0.128$ *radians/sec.*

(c) The area of the trapezoid is $a = \frac{1}{2}C(B+D)$. Notice that B and C are constants. We differentiate both sides with respect to t: $\frac{da}{dt} = \frac{1}{2}C\frac{dD}{dt}$.

Now we Plug In and solve for $\frac{da}{dt}$: $\frac{da}{dt} = \frac{1}{2}(90)(-18) = -810\ \ ft^2/\text{sec}$.

Problem 3. A body is coasting to a stop and the only force acting on it is a resistance proportional to its speed, according to the equation $\frac{ds}{dt} = v_f = v_0\, e^{-\left(\frac{k}{m}\right)t}$; $s(0)=0$, where v_0 is the body's initial velocity (in *m/s*), v_f is the its final velocity, m is its mass, k is a constant, and t is time.

(a) If a body with mass m = 50*kg* and k = 1.5*kg/sec*, initially has a velocity of 30 *m/s*, how long, to the nearest second, will it take to slow to 1 *m/s*?

(b) How far, to the 10 nearest meters, will the body coast during the time it takes to slow from 30 *m/s* to 1 *m/s*?

(c) If the body coasts from 30 *m/s* to a stop, how far will it coast?

(a) We simply Plug Into the formula and solve for t.

We get: $v_f = v_0\, e^{-\left(\frac{k}{m}\right)t}$ $1 = 30\, e^{-\left(\frac{1.5}{50}\right)t}$

Divide both sides by 30: $\frac{1}{30} = e^{-\left(\frac{1.5}{50}\right)t}$

Take the log of both sides: $\ln\frac{1}{30} = -\left(\frac{1.5}{50}\right)t$

Multiply both sides by $\frac{50}{1.5}$: $-\frac{50}{1.5}\ln\frac{1}{30} = t \approx 113$ *seconds*

(b) Now we need to solve the differential equation $\frac{ds}{dt} = v_0\, e^{-\left(\frac{k}{m}\right)t}$, which we can do with separation of variables. First, multiply both sides by dt: $ds = v_0\, e^{-\left(\frac{k}{m}\right)t} dt$

Integrate both sides: $\int ds = \int v_0\, e^{-\left(\frac{k}{m}\right)t} dt$

Evaluate the integrals: $s = -\frac{mv_0}{k} e^{-\left(\frac{k}{m}\right)t} + C$. Now Plug In the initial conditions to solve for C:

$$0 = -\frac{(50)(30)}{1.5} e^{-\left(\frac{1.5}{50}\right)(0)} + C$$

$$C = \frac{(30)(50)}{1.5} = 1000$$

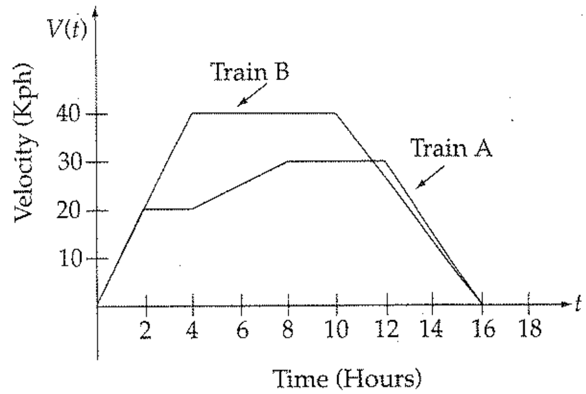
Therefore, $s = -\frac{mv_0}{k} e^{-\left(\frac{k}{m}\right)t} + 1000$. Now we Plug In the time $t = 113$ that we found in part (a) as well as the initial conditions to solve for is:

$$s = -\frac{(50)30}{1.5} e^{-\left(\frac{1.5}{50}\right)113} + 1000 \approx 970 \textit{ meters.}$$

(c) Here, because the braking force is an exponential function, the object will coast to a stop after an infinite amount of time. In other words, we need to find

$$\lim_{t\to\infty} s(t) = \lim_{t\to\infty}\left[1000 - 1000e^{-\left(\frac{k}{m}\right)t}\right] = 1000 \textit{ meters.}$$

PROBLEM 4.



Three trains, A, B, and C each travel on a straight track for $0 \le t \le 16$ hours. The graphs above, which consist of line segments, show the velocities, in kilometers per hour, of trains A and B. The velocity of C is given by $v(t) = 8t - 0.25t^2$

(Indicate units of measure for all answers.)

(a) Find the velocities of A and C at time $t = 6$ hours.

(b) Find the accelerations of B and C at time $t = 6$ hours.

(c) Find the positive difference between the total distance that A traveled and the total distance that B traveled in 16 hours.

(d) Find the total distance that C traveled in 16 hours.

(a) We can find the velocity of train A at time $t = 6$ simply by reading the graph. We get $v_A(6) = 25\ kph$. We find the velocity of train C at time $t = 6$ by plugging $t = 6$ into the formula. We get $v_C(6) = 8(6) - .25(6^2) = 39$ *kilometers per hour.*

(b) Acceleration is the derivative of velocity with respect to time. For train B, we look at the *slope* of the graph at time $t = 6$. We get $a_B(6) = 0\ km/hr^2$. For train C, we take the derivative of v. We get: $a(t) = 8 - .5t$. At time $t = 6$, we get $a_C(6) = 5\ km/hr^2$.

(c) In order to find the total distance that train A traveled in 16 hours, we need to find the area under the graph. We can find this area by adding up the areas of the different geometric objects that are under the graph. From time $t = 0$ to $t = 2$, we need to find the area of a triangle with a base of 2 and a height of 20. The area is 20. Next, from time $t = 2$ to $t = 4$, we need to find the area of a rectangle with a base of 2 and a height of 20. The area is 40. Next, from time $t = 4$ to $t = 8$, we need to find the area of a trapezoid with bases of 20 and 30 and a height of 4. The area is 100. Next, from time $t = 8$ to $t = 12$, we need to find the area of a rectangle with a base of 4 and a height of 30. The area is 120. Finally, from time $t = 12$ to $t = 16$, we need to find the area of a triangle with a base of 4 and a height of 30. The area is 60. Thus the total distance that train A traveled is 340 *km*.

Let's repeat the process for train B. From time $t = 0$ to $t = 4$, we need to find the area of a triangle with a base of 4 and a height of 40. The area is 80. Next, from time $t = 4$ to $t = 10$, we need to find the area of a rectangle with a base of 6 and a height of 40. The area is 120. Finally, from time $t = 10$ to $t = 16$, we need to find the area of a triangle with a base of 6 and a height of 40. The area is 120. Thus the total distance that train B traveled is 320 *km*.

Therefore, the positive difference between their distances is 20 *km*.

(d) First, note that the graph of train C's velocity, $v(t) = 8t - 0.25t^2$, is above the x-axis on the entire interval. Therefore, in order to find the total distance traveled, we integrate $v(t)$ over the interval. We get: $\int_0^{16} \left(8t - .25t^2\right) dt$.

Evaluate the integral: $\int_0^{16} \left(8t - .25t^2\right) dt = \left(4t^2 - \frac{t^3}{12}\right)\Big|_0^{16} = \frac{2048}{3}$ *km*.

Problem 5. Let y be the function satisfying $f'(x) = x(1 - f(x))$; $f(0) = 10$.

(a) Use Euler's method, starting at $x = 0$, with step size of 0.5 to approximate $f(x)$ at $x = 1$.

(b) Find an exact solution for $f(x)$, in terms of x, when $x = 1$.

(c) Evaluate $\int_0^{\infty} x(1 - f(x))dx$

(a) We use Euler's Method to find an approximate answer to the differential equation. First, we need a starting point, (x_0, y_0) and an initial slope, y_0'. (Note that here $f(x) = y$.) Next, we use increments of h to come up with approximations. Each new approximation will use the following rules:

$$x_n = x_{n-1} + h$$

$$y_n = y_{n-1} + h \cdot y'_{n-1}$$

Repeat for $n = 1, 2, 3, \ldots$

We are given that the curve goes through the point $(0,10)$. We will call the coordinates of this point $x_0 = 0$ and $y_0 = 10$. The slope is found by plugging these coordinates into $y' = x(1-y)$, so we have an initial slope of $y_0' = 0$.
Now we need to find the next set of points.

Step 1: Increase x_0 by h to get x_1. $x_1 = 0.5$

Step 2: Multiply h by y_0' and add to y_0 to get y_1. $y_1 = 10 + 0.5(0) = 10$

Step 3: Find y_1' by plugging x_1 and y_1 into the equation for y'

$$y_1' = (0.5)(1-10) = -4.5$$

Repeat until we get to the desired point (in this case $x = 1$).

Step 1: Increase x_1 by h to get x_2. $x_2 = 1$

Step 2: Multiply h by y_1' and add to y_1 to get y_2.

$$y_2 = 10 + 0.5(-4.5) = 7.75$$

(b) Here, we need to solve the differential equation $\frac{dy}{dx} = x(1-y)$. We can use separation of variables. Move the term containing y to the left side and the dx to the right side. We get: $\frac{dy}{(1-y)} = x\,dx$

Integrate both sides: $\int \frac{dy}{(1-y)} = \int x\,dx$

$$-\ln(1-y) = \frac{x^2}{2} + C$$

It's traditional to isolate y. First, exponentiate both sides to base e: $(1-y)=e^{-\frac{x^2}{2}+C}$

Next, use the rules of exponents to rewrite this: $(1-y)=e^{C}e^{-\frac{x^2}{2}}$. Because e^C is a constant, we can rewrite this as: $(1-y)=Ce^{-\frac{x^2}{2}}$

Finally, we isolate y: $y=1-Ce^{-\frac{x^2}{2}}$

Now, we use the initial condition to solve for C: $10 = 1 - Ce^0$. Therefore $C=-9$

Thus, because $f(x)=y$, the exact solution is: $f(x)=1+9e^{-\frac{x^2}{2}}$

Therefore, $f(1)=1+9e^{-\frac{1}{2}}$

(c) Using the Fundamental Theorem of Calculus, we know that

$\int_a^b f'(x)dx= f(b)-f(a)$. In part (b), we found that $f(x)=1+9e^{-\frac{x^2}{2}}$, so here we need to evaluate f at infinity and at zero.

We can find f at infinity using limits: $\lim\limits_{x\to\infty} f(x)=\lim\limits_{x\to\infty} 1+9e^{-\frac{x^2}{2}}=1$

We can find f at zero by Plugging In: $f(0)=1+9e^0=10$

Therefore, $\int_0^\infty x(1-f(x))dx=1-10=-9$

Problem 6: Given $f(x)=\tan^{-1}(x)$ and $g(x)=\frac{1}{1+x}$, for $|x|\le 1$.

(a) Find the fifth-degree Taylor Polynomial and general expression for $g(x)$ about $x = 0$.

(b) Given that $\frac{d}{dx}\tan^{-1}x=\frac{1}{1+x^2}$, for $|x|\le 1$, use the result of part (a) to find the fifth-degree Taylor Polynomial and general expression for $f(x)$ about $x = 0$.

(c) Use the fifth-degree Taylor Polynomial to estimate $f\left(\frac{1}{10}\right)$.

(a) The formula for a Taylor Polynomial of order n about $x = a$ is:

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \ldots ++ \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

First, we need to find the derivatives of $g(x) = \frac{1}{1+x}$:

$$g'(x) = -(1+x)^{-2}$$

$$g''(x) = 2(1+x)^{-3}$$

$$g^{(3)}(x) = -6(1+x)^{-4}$$

$$g^{(4)}(x) = 24(1+x)^{-5}$$

$$g^{(5)}(x) = 120(1+x)^{-6}$$

Next, evaluate the derivatives about $x = 0$:

$g(0) = 1$; $g'(0) = -1$; $g''(0) = 2$; $g^{(3)}(0) = -6$; $g^{(4)}(0) = 24$; $g^{(5)}(0) = 120$

Now, if we Plug In the formula for the Taylor Polynomial, we get:

$g(x) = \frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5$, and the general expression is: $g(x) = \sum_{n=0}^{\infty} (-1)^n x^n$

(b) We can find the Taylor Polynomial for $\frac{1}{1+x^2}$ by substituting x^2 for x in the formula above. We get: $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \ldots$ (actually, we don't need the last term). Then, because $\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$, if we perform term-by-term integration on the series for $\frac{1}{1+x^2}$ we will get the Taylor Polynomial for $\tan^{-1} x$.

We get: $\tan^{-1} x = \int \left(1 - x^2 + x^4\right) dx = x - \frac{x^3}{3} + \frac{x^5}{5} + C$. Because $\tan^{-1}(0) = 0$, $C = 0$, and thus the general expression for $f(x)$ is: $\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$.

(c) We can estimate $f\left(\frac{1}{10}\right)$ simply by Plugging Into the expression that we found in part (b). We get: $f\left(\frac{1}{10}\right) = \left(\frac{1}{10}\right) - \frac{\left(\frac{1}{10}\right)^3}{3} + \frac{\left(\frac{1}{10}\right)^5}{5} = \frac{1}{10} - \frac{1}{3000} + \frac{1}{500000}$, which you don't need to simplify.